

ZASSENHAUS CONJECTURE FOR INTEGRAL GROUP RING OF SIMPLE LINEAR GROUPS

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ABSTRACT. We prove that the Zassenhaus conjecture is true for $PSL(2, 8)$ and $PSL(2, 17)$. This is a continuation of research initiated by W. Kimmerle, M. Hertweck and C. Höfert.

1. INTRODUCTION

Let $U(\mathbb{Z}G)$ be the group of units of the integral group ring $\mathbb{Z}G$ of a finite group G . Then

$$U(\mathbb{Z}G) = \{\pm 1\} \times V(\mathbb{Z}G),$$

where $V(\mathbb{Z}G)$ is the group of units of augmentation one. Throughout this paper, G is always a finite group, torsion units will always represent torsion units in $V(\mathbb{Z}G) \setminus \{1\}$. One of the most important conjecture in the theory of integral group rings is

Conjecture 1. *If G is a finite group, then for each torsion unit $u \in V(\mathbb{Z}G)$ there exists $g \in G$, such that $|u| = |g|$.*

Hans Zassenhaus formulated a stronger version of this conjecture in [33], which states that:

Conjecture 2. *A torsion unit in $V(\mathbb{Z}G)$ is said to be rationally conjugate to a group element if it is conjugate to an element of G by a unit of the rational group ring $\mathbb{Q}G$.*

This conjecture was confirmed for Nilpotent Groups in [30, 32] and also some classes of solvable groups (see [20] for further details). However these techniques do not transfer to simple groups. The main investigative tool for simple group in relation to the Zassenhaus conjecture is the Luthar-Passi Method (which was introduced in [26]). It was confirmed true for all groups up to order 71 in [22]. Also this conjecture was validated for A_5 , S_5 , central extensions of S_5 and other simple finite groups in [3, 4, 26, 27]. In [31] partial results were given for A_6 , and the remaining cases were dealt with in [18]. Additionally it was proved for $PSL(2, p)$ when $p = \{7, 11, 13\}$ in [19] and further results regarding $PSL(2, p)$ can be found in [21].

Let H be a group with torsion part $t(H)$ of finite exponent. Let $\#H$ be the set of primes dividing the order of elements from $t(H)$. The prime graph of H (denoted by $\pi(H)$) is a graph with vertices labeled by primes from $\#H$, such that vertices p and q are adjacent if and only if there is an element of order pq in the group. The following question which was composed as a problem in [29] (Problem 37):

Question 1. *(Prime Graph Question) If G is a finite group, then $\pi(G) = \pi(V(\mathbb{Z}G))$.*

It was established in [24] that this question is upheld for Frobenius and Solvable groups. It was also confirmed for some Sporadic Simple groups in [2, 5–15]. The prime graph question is an intermediate step towards the Zassenhaus conjecture.

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We combine the Luthar-Passi Method together with techniques developed in [17, 19] to obtain our results. Our main results are the following:

Theorem 1. *Let $G = PSL(2, 8)$. If u be a torsion unit of $V(\mathbb{Z}G)$, then u is rationally conjugated to some element $g \in G$. Hence the Zassenhaus Conjecture is true for G .*

Theorem 2. *Let $G = PSL(2, 17)$. If u be a torsion unit of $V(\mathbb{Z}G)$, then u is rationally conjugated to some element $g \in G$. Hence the Zassenhaus Conjecture is true for G .*

Let $u = \sum a_g g$ be a torsion unit of $V(\mathbb{Z}G)$, then the order of u divides the exponent of the group (see [16]). For an element $x \in G$, let $\sum_{g \in X^G} a_g$ be the partial augmentation (denoted by $\varepsilon_C(u)$) of u with respect to it's conjugacy classes in G . Let $\nu_i = \varepsilon_{C_i}(u)$ be the i -th partial augmentation of u . G. Higman and S. D. Berman [1] proved that $\nu_1 = 0$ and $\nu_j = 0$ if the conjugacy class C_j consists of a central elements. Therefore for any torsion u (since $u \in V(\mathbb{Z}G)$), we can see that

$$\nu_2 + \nu_3 + \cdots + \nu_l = 1$$

where l denotes the class number of G .

Proposition 1. ([16]) *Let u be torsion unit of $V(\mathbb{Z}G)$. The order of u divides the exponent of G .*

The following Propositions provide relationships between the partial augmentations and the order of a torsion unit.

Proposition 2. (Proposition 3.1 in [17]) *Let u be a torsion unit of $V(\mathbb{Z}G)$. Let C be a conjugacy class of G . If p is a prime dividing the order of a representative of C but not the order of u then the partial augmentation $\varepsilon_C(u) = 0$.*

Proposition 3. (Proposition 2.2 in [19]) *Let G be a finite group and let u be a torsion unit in $V(\mathbb{Z}G)$.*

- (i) *If u has order p^n , then $\varepsilon_x(u) = 0$ for every x of G whose p -part is of order strictly greater than p^n .*
- (ii) *If x is an element of G whose p -part, for some prime, has order strictly greater than the order of the p -part of u , then $\varepsilon_x(u) = 0$.*

Proposition 4. ([26] and Theorem 2.5 in [28]) *Let u be a torsion unit of $V(\mathbb{Z}G)$ of order k . Then u is conjugate in $\mathbb{Q}G$ to an element $g \in G$ iff for each d dividing k there is precisely one conjugacy class C_{i_d} with partial augmentation $\varepsilon_{C_{i_d}}(u^d) \neq 0$.*

Proposition 5. (Proposition 6.1 in [19]) *Let $G = PSL(2, p^f)$ for an odd prime p , and $f \leq 2$. Then units of order p in $V(\mathbb{Z}G)$ are rationally conjugate to elements of G .*

Proposition 6. (Proposition 6.3 in [19]) *Let $G = PSL(2, p)$ for an odd prime p . If the order of a torsion unit u in $V(\mathbb{Z}G)$ is divisible by p , then u is of order p .*

Proposition 7. (Proposition 6.4 in [19]) *Let $G = PSL(2, p^f)$, and let r be a prime different from p . Then any torsion unit u in $V(\mathbb{Z}G)$ of order r is rationally conjugate to an element of G .*

Proposition 8. (Proposition 6.6 in [19]) *Let $G = PSL(2, p^f)$ with $p \neq 2, 3$. Then any torsion unit in $V(\mathbb{Z}G)$ of order 6 is rationally conjugate to an element of G .*

Proposition 9. (Proposition 6.7 in [19]) *Let $G = PSL(2, p^f)$. Then for any p -regular torsion unit u in $V(\mathbb{Z}G)$, there is an element of G of the same order as u .*

For any character χ of G and any torsion unit of $V(\mathbb{Z}G)$, clearly $\chi(u) = \sum_{i=1}^l \nu_i \chi(h_i)$ where h_i is a representative of a conjugacy class C_i .

Proposition 10. (Theorem 1 in [26] and [19]) *Let p be equal to zero or a prime divisor of $|G|$. Suppose that u is an element of $V(\mathbb{Z}G)$ of order k . Let z be a*

primitive k -th root of unity. Then for every integer l and any character χ of G , the number

$$\mu_l(u, \chi, p) = \frac{1}{k} \sum_{d|k} \text{Tr}_{\mathbb{Q}(z^d)/\mathbb{Q}} \{ \chi(u^d) z^{-dl} \}$$

is a non-negative integer.

We will use the notation $\mu_l(u, \chi, *)$ when $p = 0$. The LAGUNA package [25] for the GAP system [23] is a very useful tool when calculating $\mu_l(u, \chi, p)$.

2. PROOF OF THEOREM 1

Let $G = PSL(2, 8)$. Clearly $|G| = 504 = 2^3 \cdot 3^2 \cdot 7$ and $\exp(G) = 126 = 2 \cdot 3^2 \cdot 7$. Initially for any torsion unit of $V(\mathbb{Z}G)$ of order k , we have that

$$\nu_{2a} + \nu_{3a} + \nu_{7a} + \nu_{7b} + \nu_{7c} + \nu_{9a} + \nu_{9b} + \nu_{9c} = 1.$$

By Proposition 1, we need only to consider torsion units of $V(\mathbb{Z}G)$ of order 2, 3, 7, 9, 6, 14, 21.

Case (i). Let u be a torsion unit in $V(\mathbb{Z}G)$ of order 2. By Proposition 2, $\nu_{kx} = 0 \ \forall \ kx \in \{3a, 7a, 7b, 7c, 9a, 9b, 9c\}$. Therefore u is rationally conjugated to some element $g \in G$ by Proposition 4.

Case (ii). Let u be a torsion unit in $V(\mathbb{Z}G)$ of order 3 and 7. Clearly u is rationally conjugated to some element $g \in G$ by Proposition 7.

It remains only to consider torsion units of order 6, 9, 14 and 21.

Case (iii). Let u be a torsion unit in $V(\mathbb{Z}G)$ of order 6. Using Propositions 2 & 3, $\nu_{2a} + \nu_{3a} = 1$. Now $\chi(u^3) = \chi(2a)$ and $\chi(u^2) = \chi(3a)$. Applying Proposition 10 to the character table (Table 1), we obtain

$$\begin{aligned} \mu_3(u, \chi_2, *) &= \frac{1}{6}(2\gamma_1 + 4) \geq 0; & \mu_0(u, \chi_2, *) &= \frac{1}{6}(-2\gamma_1 + 2) \geq 0; \\ \mu_1(u, \chi_2, *) &= \frac{1}{6}(-\gamma_1 + 10) \geq 0; & \mu_0(u, \chi_3, *) &= \frac{1}{6}(-2\gamma_2 + 8) \geq 0, \end{aligned}$$

where $\gamma_1 = \nu_{2a} + 2\nu_{3a}$ and $\gamma_2 = \nu_{2a} - \nu_{3a}$. Clearly $\gamma_1 = -2$. It follows that there are no possible integer solutions for (ν_{2a}, ν_{3a}) .

Case (iv). Let u be a torsion unit in $V(\mathbb{Z}G)$ of order 9. Using Propositions 2 & 3, $\nu_{3a} + \nu_{9a} + \nu_{9b} + \nu_{9c} = 1$. Now $\chi(u^3) = \chi(3a)$. Applying Proposition 10 to the character tables (Tables 1, 2 & 3), we obtain

$$\begin{aligned} \mu_2(u, \chi_2, 7) &= \frac{1}{9}(3\gamma_1 + 3) \geq 0; & \mu_0(u, \chi_2, 7) &= \frac{1}{9}(-6\gamma_1 + 3) \geq 0; \\ \mu_1(u, \chi_2, 2) &= \frac{1}{9}(3\gamma_2 + 3) \geq 0; & \mu_1(u, \chi_3, *) &= \frac{1}{9}(-3\gamma_2 + 6) \geq 0; \\ \mu_2(u, \chi_3, *) &= \frac{1}{9}(3\gamma_3 + 6) \geq 0; & \mu_2(u, \chi_2, 2) &= \frac{1}{9}(-3\gamma_3 + 3) \geq 0; \\ \mu_4(u, \chi_2, 2) &= \frac{1}{9}(-3\gamma_4 + 3) \geq 0, \end{aligned}$$

where $\gamma_1 = 2\nu_{3a} - \nu_{9a} - \nu_{9b} - \nu_{9c}$, $\gamma_2 = 2\nu_{9a} - \nu_{9b} - \nu_{9c}$, $\gamma_3 = \nu_{9a} - 2\nu_{9b} + \nu_{9c}$ and $\gamma_4 = \nu_{9a} + \nu_{9b} - 2\nu_{9c}$. Clearly $\gamma_1 = -1$, $\gamma_2 \in \{-1, 2\}$ and $\gamma_3 \in \{-2, 1\}$. It follows that the only possible integer values for $(\nu_{3a}, \nu_{9a}, \nu_{9b}, \nu_{9c})$ are $(0, 1, 0, 0)$, $(0, 0, 1, 0)$ and $(0, 0, 0, 1)$. Therefore u is rationally conjugated to some element $g \in G$ by Proposition 4.

Case (v). Let u be a torsion unit in $V(\mathbb{Z}G)$ of order 14. Using Propositions 2 & 3, $\nu_{2a} + \nu_{7a} + \nu_{7b} + \nu_{7c} = 1$. Now $\chi(u^7) = \chi(2a)$ and $\chi(u^2) = \chi(7a)$. Applying Proposition 10 to the character table (Table 1), we obtain

$$\begin{aligned} \mu_7(u, \chi_2, *) &= \frac{1}{14}(6\nu_{2a} + 8) \geq 0; & \mu_0(u, \chi_2, *) &= \frac{1}{14}(-6\nu_{2a} + 6) \geq 0; \\ \mu_1(u, \chi_2, *) &= \frac{1}{14}(-\nu_{2a} + 8) \geq 0. \end{aligned}$$

Clearly $\nu_{2a} = 1$. It follows that there are no possible integer solutions for ν_{2a} . The exact same inequalities are obtained when $\chi(u^7) = \chi(2a)$ and $\chi(u^2) = \chi(7b)$, and also when $\chi(u^7) = \chi(2a)$ and $\chi(u^2) = \chi(7c)$.

Case (vi). Let u be a torsion unit in $V(\mathbb{Z}G)$ of order 21. Clearly u is rationally conjugated to some element $g \in G$ by Proposition 9. This completes the proof.

TABLE 1. Character Table of $PSL(2, 8)$ ($p = 0$, [23])

	1a	2a	3a	7a	7b	7c	9a	9b	9c
χ_1	1	1	1	1	1	1	1	1	1
χ_2	7	-1	-2	0	0	0	1	1	1
χ_3	7	-1	1	0	0	0	\mathcal{D}	\mathcal{E}	\mathcal{F}
χ_4	7	-1	1	0	0	0	\mathcal{E}	\mathcal{F}	\mathcal{D}
χ_5	7	-1	1	0	0	0	\mathcal{F}	\mathcal{D}	\mathcal{E}
χ_6	8	0	-1	1	1	1	-1	-1	-1
χ_7	9	1	0	\mathcal{A}	\mathcal{C}	\mathcal{B}	0	0	0
χ_8	9	1	0	\mathcal{B}	\mathcal{A}	\mathcal{C}	0	0	0
χ_9	9	1	0	\mathcal{C}	\mathcal{B}	\mathcal{A}	0	0	0

where $\mathcal{A} = \alpha^3 + \alpha^4$, $\mathcal{B} = \alpha^2 + \alpha^5$, $\mathcal{C} = \alpha + \alpha^6$, $\mathcal{D} = -\zeta^4 - \zeta^5$, $\mathcal{E} = -\zeta^2 - \zeta^7$, $\mathcal{F} = \zeta^2 + \zeta^4 + \zeta^5 + \zeta^7$, $\alpha = e^{\frac{2\pi i}{7}}$ and $\zeta = e^{\frac{2\pi i}{9}}$.

TABLE 2. Character Table of $PSL(2, 8)$ ($p = 2$, [23])

	1a	3a	7a	7b	7c	9a	9b	9c
χ_1	1	1	1	1	1	1	1	1
χ_2	2	-1	\mathcal{A}	\mathcal{C}	\mathcal{B}	\mathcal{G}	\mathcal{I}	\mathcal{H}
χ_3	2	-1	\mathcal{B}	\mathcal{A}	\mathcal{C}	\mathcal{H}	\mathcal{G}	\mathcal{I}
χ_4	2	-1	\mathcal{C}	\mathcal{B}	\mathcal{A}	\mathcal{I}	\mathcal{H}	\mathcal{G}
χ_5	4	1	\mathcal{D}	\mathcal{F}	\mathcal{E}	\mathcal{J}	\mathcal{L}	\mathcal{K}
χ_6	4	1	\mathcal{E}	\mathcal{D}	\mathcal{F}	\mathcal{K}	\mathcal{J}	\mathcal{L}
χ_7	4	1	\mathcal{F}	\mathcal{E}	\mathcal{D}	\mathcal{L}	\mathcal{K}	\mathcal{J}
χ_8	8	-1	1	1	1	-1	-1	-1

where $\mathcal{A} = \alpha + \alpha^6$, $\mathcal{B} = \alpha^3 + \alpha^4$, $\mathcal{C} = \alpha^2 + \alpha^5$, $\mathcal{D} = \alpha + \alpha^3 + \alpha^4 + \alpha^6$, $\mathcal{E} = \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5$, $\mathcal{F} = \alpha + \alpha^2 + \alpha^5 + \alpha^6$, $\mathcal{G} = -\zeta^2 - \zeta^4 - \zeta^5 - \zeta^7$, $\mathcal{H} = \zeta^4 + \zeta^5$, $\mathcal{I} = \zeta^2 + \zeta^7$, $\mathcal{J} = -\zeta^2 + \zeta^3 - \zeta^4 - \zeta^5 + \zeta^6 - \zeta^7$, $\mathcal{K} = \zeta^3 + \zeta^4 + \zeta^5 + \zeta^6$, $\mathcal{L} = \zeta^2 + \zeta^3 + \zeta^6 + \zeta^7$, $\alpha = e^{\frac{2\pi i}{7}}$ and $\zeta = e^{\frac{2\pi i}{9}}$.

TABLE 3. Character Table of $PSL(2, 8)$ ($p = 7$, [23])

	1a	2a	3a	9a	9b	9c
χ_1	1	1	1	1	1	1
χ_2	7	-1	-2	1	1	1
χ_3	7	-1	1	\mathcal{A}	\mathcal{C}	\mathcal{B}
χ_4	7	-1	1	\mathcal{B}	\mathcal{A}	\mathcal{C}
χ_5	7	-1	1	\mathcal{C}	\mathcal{B}	\mathcal{A}
χ_6	8	0	-1	-1	-1	-1

where $\mathcal{A} = \zeta^2 + \zeta^4 + \zeta^5 + \zeta^7$, $\mathcal{B} = -\zeta^4 - \zeta^5$, $\mathcal{C} = -\zeta^2 - \zeta^7$ and $\zeta = e^{\frac{2\pi i}{9}}$.

3. PROOF OF THEOREM 2

Let $G = PSL(2, 17)$. Clearly $|G| = 2448 = 2^4 \cdot 3^2 \cdot 17$ and $\exp(G) = 1224 = 2^3 \cdot 3^2 \cdot 17$. Initially for any torsion unit of $V(\mathbb{Z}G)$ of order k , we have that

$$\nu_{2a} + \nu_{3a} + \nu_{4a} + \nu_{8a} + \nu_{8b} + \nu_{9a} + \nu_{9b} + \nu_{9c} + \nu_{17a} + \nu_{17b} = 1.$$

By Proposition 1, we need only to consider torsion units of $V(\mathbb{Z}G)$ of order 2, 3, 4, 6, 8, 9, 17, 34 and 51.

Case (i). Let u be a torsion unit in $V(\mathbb{Z}G)$ of order 2 and 3. Clearly u is rationally conjugated to some element $g \in G$ by Propositions 2 and 4.

Case (ii). Let u be a torsion unit in $V(\mathbb{Z}G)$ of order 17. Using Proposition 5, u is rationally conjugated to some element $g \in G$.

Case (iii). Let u be a torsion unit in $V(\mathbb{Z}G)$ of order 4. Using Propositions 2 & 3, $\nu_{2a} + \nu_{4a} = 1$. Applying Proposition 10 to the character table (Table 5), we obtain

$$\mu_2(u, \chi_2, 17) = \frac{1}{4}(2\gamma_1 + 2) \geq 0; \quad \mu_0(u, \chi_2, 17) = \frac{1}{4}(-2\gamma_1 + 2) \geq 0,$$

where $\gamma_1 = \nu_{2a} - \nu_{4a}$. Clearly $\gamma_1 \in \{-1, 1\}$. It follows that the only possible integer values for (ν_{2a}, ν_{4a}) are $(1, 0)$ and $(0, 1)$. Therefore u is rationally conjugated to some element $g \in G$ by Proposition 4.

Case (iv). Let u be a torsion unit in $V(\mathbb{Z}G)$ of order 8. Using Propositions 2 & 3,

$$\nu_{2a} + \nu_{4a} + \nu_{8a} + \nu_{8b} = 1.$$

Applying Proposition 10 to the character tables (Table 4 & 5), we obtain

$$\begin{aligned} \mu_4(u, \chi_4, 17) &= \frac{1}{8}(4\gamma_1 + 4) \geq 0; & \mu_0(u, \chi_4, 17) &= \frac{1}{8}(-4\gamma_1 + 4) \geq 0; \\ \mu_0(u, \chi_9, *) &= \frac{1}{8}(8\gamma_2 + 16) \geq 0; & \mu_4(u, \chi_9, *) &= \frac{1}{8}(-8\gamma_2 + 16) \geq 0; \\ \mu_1(u, \chi_2, 17) &= \frac{1}{8}(4\gamma_3 + 4) \geq 0; & \mu_3(u, \chi_2, 17) &= \frac{1}{8}(-4\gamma_3 + 4) \geq 0; \\ \mu_4(u, \chi_2, 17) &= \frac{1}{8}(4\gamma_4 + 4) \geq 0; & \mu_4(u, \chi_3, 17) &= \frac{1}{8}(4\gamma_5 + 4) \geq 0, \end{aligned}$$

where $\gamma_1 = \nu_{2a} + \nu_{4a} - \nu_{8a} - \nu_{8b}$, $\gamma_2 = \nu_{2a} - \nu_{4a}$ and $\gamma_3 = \nu_{8a} - \nu_{8b}$, $\gamma_4 = \nu_{2a} - \nu_{4a} - \nu_{8a} - \nu_{8b}$ and $\gamma_5 = -\nu_{2a} + \nu_{4a} - \nu_{8a} - \nu_{8b}$. Now $\gamma_1 \in \{-1, 1\}$, $\gamma_2 \in \{-2, -1, 0, 1, 2\}$, $\gamma_3 \in \{-1, 1\}$. It follows that the only possible integer values for $(\nu_{2a}, \nu_{4a}, \nu_{8a}, \nu_{8b})$ are $(0, 0, 0, 1)$ and $(0, 0, 1, 0)$. Therefore u is rationally conjugated to some element $g \in G$ by Proposition 4.

Case (v). Let $|u| = 9$. Using Propositions 2 & 3,

$$\nu_{3a} + \nu_{9a} + \nu_{9b} + \nu_{9c} = 1.$$

Applying Proposition 10 to the character tables (Table 4 & 5), we obtain

$$\begin{aligned} \mu_3(u, \chi_4, *) &= \frac{1}{9}(3\gamma_1 + 12) \geq 0; & \mu_0(u, \chi_4, *) &= \frac{1}{9}(-6\gamma_1 + 12) \geq 0; \\ \mu_2(u, \chi_2, 17) &= \frac{1}{9}(3\gamma_2 + 3) \geq 0; & \mu_1(u, \chi_3, 17) &= \frac{1}{9}(-3\gamma_2 + 6) \geq 0; \\ \mu_2(u, \chi_3, 17) &= \frac{1}{9}(3\gamma_3 + 6) \geq 0; & \mu_4(u, \chi_2, 17) &= \frac{1}{9}(-3\gamma_3 + 3) \geq 0; \\ \mu_1(u, \chi_2, 17) &= \frac{1}{9}(-3\gamma_4 + 3) \geq 0, \end{aligned}$$

where $\gamma_1 = 2\nu_{3a} - \nu_{9a} - \nu_{9b} - \nu_{9c}$, $\gamma_2 = 2\nu_{9a} - \nu_{9b} - \nu_{9c}$, $\gamma_3 = \nu_{9a} - 2\nu_{9b} + \nu_{9c}$ and $\gamma_4 = \nu_{9a} + \nu_{9b} - 2\nu_{9c}$. Now $\gamma_1 \in \{-4, -1, 2\}$, $\gamma_2 \in \{-1, 2\}$ and $\gamma_3 \in \{-2, 1\}$. It follows that the only possible integer values for $(\nu_{3a}, \nu_{9a}, \nu_{9b}, \nu_{9c})$ are $(0, 0, 0, 1)$, $(0, 0, 1, 0)$ and $(0, 1, 0, 0)$. Therefore u is rationally conjugated to some element $g \in G$ by Proposition 4.

Case (vi). Let us consider all possible values of $|u|$ which do not appear in G . By [16], $|u| \in \{6, 34, 51\}$. By proposition 8, there doesn't exist any torsion units of order 6 (since G doesn't contain any such elements). Finally if $|u| \in \{34, 51\}$, then

u is rationally conjugated to some element $g \in G$ by Proposition 6. This completes the proof.

TABLE 4. Character Table of $PSL(2, 17)$ ($p = 0$, [23])

	1a	2a	3a	4a	8a	8b	9a	9b	9c	17a	17b
χ_1	1	1	1	1	1	1	1	1	1	1	1
χ_2	9	1	0	1	-1	-1	0	0	0	\mathcal{E}	\mathcal{F}
χ_3	9	1	0	1	-1	-1	0	0	0	\mathcal{F}	\mathcal{E}
χ_4	16	0	-2	0	0	0	1	1	1	-1	-1
χ_5	16	0	1	0	0	0	\mathcal{B}	\mathcal{C}	\mathcal{D}	-1	-1
χ_6	16	0	1	0	0	0	\mathcal{C}	\mathcal{D}	\mathcal{B}	-1	-1
χ_7	16	0	1	0	0	0	\mathcal{D}	\mathcal{B}	\mathcal{C}	-1	-1
χ_8	17	1	-1	1	1	1	-1	-1	-1	0	0
χ_9	18	2	0	-2	0	0	0	0	0	1	1
χ_{10}	18	-2	0	0	\mathcal{A}	$-\mathcal{A}$	0	0	0	1	1
χ_{11}	18	-2	0	0	$-\mathcal{A}$	\mathcal{A}	0	0	0	1	1

where $\mathcal{A} = -\alpha + \alpha^3$, $\mathcal{B} = -\zeta^2 - \zeta^7$, $\mathcal{C} = -\zeta^4 - \zeta^5$, $\mathcal{D} = \zeta^2 + \zeta^4 + \zeta^5 + \zeta^7$,
 $\mathcal{E} = -\delta - \delta^2 - \delta^4 - \delta^8 - \delta^9 - \delta^{13} - \delta^{15} - \delta^{16}$,
 $\mathcal{F} = -\delta^3 - \delta^5 - \delta^6 - \delta^7 - \delta^{10} - \delta^{11} - \delta^{12} - \delta^{14}$, $\alpha = e^{\frac{2\pi i}{8}}$, $\zeta = e^{\frac{2\pi i}{9}}$ and $\delta = e^{\frac{2\pi i}{17}}$.

TABLE 5. Character Table of $PSL(2, 17)$ ($p = 17$, [23])

	1a	2a	3a	4a	8a	8b	9a	9b	9c
χ_1	1	1	1	1	1	1	1	1	1
χ_2	3	-1	0	1	\mathcal{A}	\mathcal{K}	\mathcal{B}	\mathcal{E}	\mathcal{H}
χ_3	5	1	-1	-1	\mathcal{A}	\mathcal{K}	\mathcal{C}	\mathcal{F}	\mathcal{I}
χ_4	7	-1	1	-1	1	1	\mathcal{D}	\mathcal{G}	\mathcal{J}
χ_5	9	1	0	1	-1	-1	0	0	0
χ_6	11	-1	-1	1	$-\mathcal{A}$	$-\mathcal{K}$	$-\mathcal{D}$	$-\mathcal{G}$	$-\mathcal{J}$
χ_7	13	1	1	-1	$-\mathcal{A}$	$-\mathcal{K}$	$-\mathcal{C}$	$-\mathcal{F}$	$-\mathcal{I}$
χ_8	15	-1	0	-1	-1	-1	$-\mathcal{B}$	$-\mathcal{E}$	$-\mathcal{H}$
χ_9	17	1	-1	1	1	1	-1	-1	-1

where $\mathcal{A} = 1 + \alpha - \alpha^3$, $\mathcal{B} = \zeta^2 - \zeta^3 - \zeta^6 + \zeta^7$, $\mathcal{C} = \zeta^2 - \zeta^3 + \zeta^4 + \zeta^5 - \zeta^6 + \zeta^7$,
 $\mathcal{D} = \zeta^2 + \zeta^4 + \zeta^5 + \zeta^7$, $\mathcal{E} = -\zeta^3 + \zeta^4 + \zeta^5 - \zeta^6$, $\mathcal{F} = -\zeta^2 - \zeta^3 - \zeta^6 - \zeta^7$,
 $\mathcal{G} = -\zeta^2 - \zeta^7$, $\mathcal{H} = -\zeta^2 - \zeta^3 - \zeta^4 - \zeta^5 - \zeta^6 - \zeta^7$, $\mathcal{I} = -\zeta^3 - \zeta^4 - \zeta^5 - \zeta^6$,
 $\mathcal{J} = -\zeta^4 - \zeta^5$, $\mathcal{K} = 1 - \alpha + \alpha^3$, $\alpha = e^{\frac{2\pi i}{8}}$ and $\zeta = e^{\frac{2\pi i}{9}}$.

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